Multiple Choice

1 C) $\overrightarrow{PQ}=\overrightarrow{PO}+\overrightarrow{OQ}=\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}5\\4\end{pmatrix}$.

2 B) $\int \sin^2 3x dx=\int \frac{1-\cos 6x}{2}dx$.

3 D) $P(-2)=8-8+6+8=14$.

4 A) $\int ydy=\int xdx, \therefore y^2=x^2+c$.

5 B) $\overrightarrow{OA}.\overrightarrow{OB}=OA\times OB\cos\theta<0$ when $\cos\theta<0$.

6 A) $P(X\ge 1)=1-P(X=0)=1-0.1^{10}=0.\dot{9}$.

7 D) $-a\sin x+b\cos x=-R\sin\left(x-\tan^{-1}\frac{b}{a}\right)$,

$\therefore$ it's the graph of $R\sin x$ moved to the right a little $\left(=\tan^{-1}\frac{b}{a}\right)$ then flipped about the x-axis.

8 C) For $-\frac{\pi}{2}\le t\le\frac{\pi}{2}, -1\le \sin x\le 1, 0\le\cos x\le 1$. The semi-circle has domain $2\le x\le 4$, range $1\le y\le 2$. $\therefore$ (C).

9 A) $\sin^{-1}(\sin x)=x$ for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, and $-1\le\sin^{-1}A\le 1$

10 C) 15×236 votes $=3540$. Thus, the remaining 3 votes can be shared between 2 candidates, i.e. the winner receives 2 additional votes.

Question 11

(a) $\left(\underset{\sim}{i}+6\underset{\sim}{j}\right)+\left(2\underset{\sim}{i}-7\underset{\sim}{j}\right)=\left(3\underset{\sim}{i}-\underset{\sim}{j}\right)$.

(b) $(2a-b)^4=16a^4-4\times 8a^3b+6\times 4a^2b^2-4\times 2ab^3+b^4$

$=16a^4-32a^3b+24a^2b^2-8ab^3+b^4$.

(c) Let $u=x+1, x=u-1, dx=du$.

$$\begin{aligned}\int x\sqrt{x+1}dx &= \int (u-1)\sqrt{u}du\\ &=\int\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du\\ &=\frac{2u^{\frac{5}{2}}}{5}-\frac{2u^{\frac{3}{2}}}{3}+c\\ &=\frac{2(x+1)^2\sqrt{x+1}}{5}-\frac{2(x+1)\sqrt{x+1}}{3}+c.\end{aligned}$$

(d) ${}^{10}C_5\times{}^{8}C_3=14\ 112$.

(e) $V=\frac{4}{3}\pi r^3$,

$$\begin{aligned}\therefore \frac{dV}{dt}&=4\pi r^2\frac{dr}{dt}\\ &=4\pi\times 0.6^2\times 0.2\\ &=0.9\text{ mm}^3/\text{s}.\end{aligned}$$

(f) $\int_0^{\sqrt{3}}\frac{1}{\sqrt{4-x^2}}dx=\left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{5}}$

$=\frac{\pi}{3}$.

(g) $$\begin{aligned}2\sin^3 x+2\sin^2 x-\sin x-1 &=2\sin^2 x(\sin x+1)-(\sin x+1)\\ &=\left(2\sin^2 x-1\right)(\sin x+1)\\ &=0\text{ when }\sin x=\pm\frac{1}{\sqrt{2}},-1.\end{aligned}$$

$\sin x=\frac{1}{\sqrt{2}}, \therefore x=\frac{\pi}{4},\frac{3\pi}{4}$

$\sin x=-\frac{1}{\sqrt{2}}, \therefore x=\frac{5\pi}{4},\frac{7\pi}{4}$

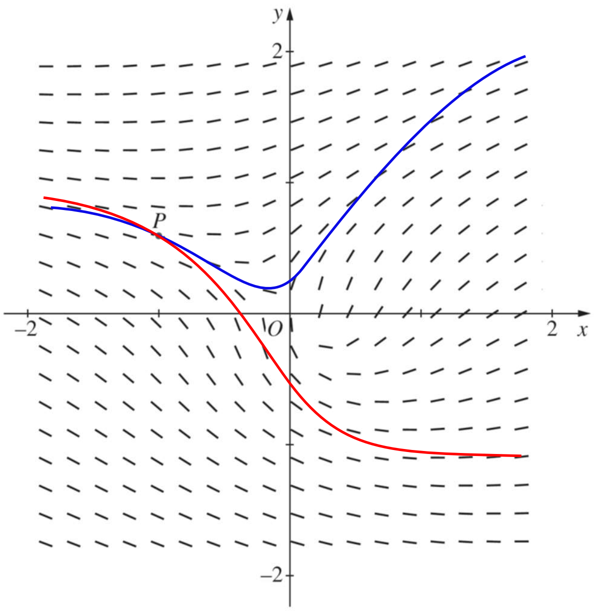
$\sin x=-1, x=\frac{3\pi}{2}$.

$\therefore$ For $0\le x\le 2\pi, x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{3\pi}{2},\frac{7\pi}{4}$.

(h) $$\begin{aligned}\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}&=\frac{\sum\alpha\beta\gamma}{\prod\alpha}\\ &=\frac{-3}{-6}\\ &=\frac{1}{2}.\end{aligned}$$

Question 12

(a) Two possible solutions:



(b) (i) $\int \frac{dT}{T-25} = k\int dt.$

$\ln\frac{T-25}{A} = kt.$

$T = 25 + Ae^{kt}.$

When $t=0, T=5, \therefore A=-20.$

When $t=8, T=10, \therefore 20e^{8t}=15, \therefore k=\frac{1}{8}\ln\frac{3}{4}.$

$\therefore T = 25-20e^{\frac{1}{8}\ln\frac{3}{4}t}.$

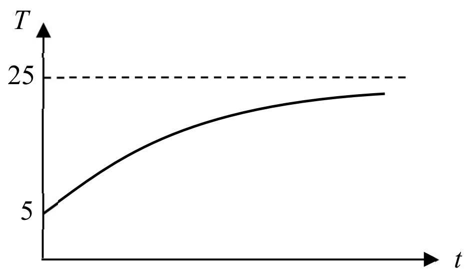
When $T=20, 20=25-20e^{\frac{1}{8}\ln\frac{3}{4}t}.$

$e^{\frac{1}{8}\ln\frac{3}{4}t} = \frac{1}{4}.$

$\frac{1}{8}\ln\frac{3}{4}t = \ln\frac{1}{4}.$

$t = 8\frac{\ln\frac{1}{4}}{\ln\frac{3}{4}} = 38.55 = 39$ minutes.

(ii)



(c) Let $n=1$, $\text{LHS} = \frac{1}{1\times2\times3} = \frac{1}{6},$

$\text{RHS} = \frac{1}{4} - \frac{1}{2\times2\times3} = \frac{1}{4}-\frac{1}{12} = \frac{1}{6}.$

$\therefore$ LHS = RHS. $\therefore$ True for $n=1$.

Assume $\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\ldots+\frac{1}{n(n+1)(n+2)}$

$=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$ for some value of n.

RTP $\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\ldots+\frac{1}{n(n+1)(n+2)}$

$+\frac{1}{(n+1)(n+2)(n+3)} = \frac{1}{4}-\frac{1}{2(n+2)(n+3)}.$

$\text{LHS} = \frac{1}{4}-\frac{1}{2(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}$

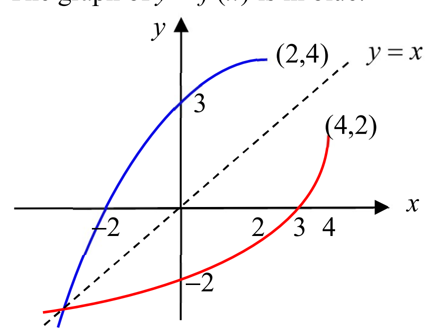
$=\frac{1}{4}-\frac{(n+3)-2}{2(n+1)(n+2)(n+3)}$

$=\frac{1}{4}-\frac{(n+1)}{2(n+1)(n+2)(n+3)}$

$=\frac{1}{4}-\frac{1}{2(n+2)(n+3)} = \text{RHS}.$

$\therefore$ The statement is true for all $n \ge 1$ by the principle of Induction.

(d) (i) The graph of $y=f(x)$ is in blue.



(ii) $f: y = 4-\left(1-\frac{x}{2}\right)^2, x\le 2, y\le 4.$

$f^{-1}: x = 4-\left(1-\frac{y}{2}\right)^2, y\le 2, x \le 4.$

$\left(1-\frac{y}{2}\right)^2 = \left(\frac{y}{2}-1\right)^2 = 4-x.$

$\frac{y}{2}-1 = \pm\sqrt{4-x},$

$y = 2\left(1-\sqrt{4-x}\right)$, taking the negative as $y \le 2$.

Domain: $x \le 4$.

(iii) The graph of $y = f^{-1}(x)$ is shown above in red.

Question 13

(a) $V=\pi\int_0^2 y\,dy-\pi\int_1^2 (y-1)dy$, using $\pi\int x^2 dy$

$=\pi\left[\frac{y^2}{2}\right]_0^2-\pi\left[\frac{(y-1)^2}{2}\right]_1^2$

$=2\pi-\frac{\pi}{2}=\frac{3\pi}{2}$ units^3.

(b) $\begin{cases} x=12t\cos 30^\circ=6\sqrt{3}t \\ y=-5t^2+12t\sin 60^\circ+1=-5t^2+6t+1 \end{cases}$

Maximum height occurs when $\dot{y}=-10t+6=0$,

$\therefore$ When $t=\frac{3}{5}$ s, $y=-5\times\frac{9}{25}+6\times\frac{3}{5}+1=2.8$ m <3 m.

$\therefore$ It does not hit the ceiling.

When $t=2\times\frac{3}{5}$ s, $x=6\sqrt{3}\times\frac{6}{5}\simeq 12.4$ m >10 m.

$\therefore$ It will hit the far wall without hitting the floor.

(c) Area $=\left|2\int_0^2\left(1-\frac{8}{4+x^2}\right)dx\right|+$ area of the 2 triangles,

$=2\left[x-4\tan^{-1}\frac{x}{2}\right]_2^0+4$

$=2\left(-2+4\tan^{-1}1\right)+4$

$=2\pi$ units^2.

(d) (i) $\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{\sin(B-d)+\sin(B+d)}{\cos(B-d)+\cos(B+d)}$

$=\frac{\sin B\cos d-\cos B\sin d+\sin B\cos d+\cos B\sin d}{\cos B\cos d+\sin B\sin d+\cos B\cos d-\sin B\sin d}$

$=\frac{2\sin B\cos d}{2\cos B\cos d}$

$=\tan B$.

(ii) Adding $A=B-d$ and $C=B+d$ gives $B=\frac{A+C}{2}$

$\therefore \frac{\sin\frac{5\theta}{7}+\sin\frac{6\theta}{7}}{\cos\frac{5\theta}{7}+\cos\frac{6\theta}{7}}=\tan\frac{11\theta}{14}$.

Since $\sqrt{3}=\tan\frac{\pi}{3}$, solving $\tan\frac{11\theta}{14}=\tan\frac{\pi}{3}$ gives

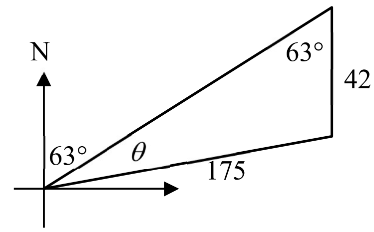
$\frac{11\theta}{14}=\frac{\pi}{3}+k\pi$.

$\theta=\frac{14\pi(1+3k)}{33}, k\in J$.

$\therefore \theta=\frac{14\pi}{33},\frac{56\pi}{33}$ for $0\le\theta\le 2\pi$.

Question 14

(a)



$\frac{\sin\theta}{42}=\frac{\sin 63^\circ}{175}$.

$\sin\theta=0.2138$ (to 4 d.p.).

$\therefore \theta\simeq 12^\circ$.

$63^\circ+12^\circ=75^\circ, \therefore$ The bearing is 075°T.

(b) $\int_{150000}^{600000}\frac{C}{P(C-P)}dP=0.1\int_0^{20}dt$.

$\int_{150000}^{600000}\left(\frac{1}{P}+\frac{1}{C-P}\right)dP=0.1\int_0^{20}dt$.

$\left[\ln\frac{P}{C-P}\right]_{150000}^{600000}=2$.

$\ln\frac{600000}{C-600000}\times\frac{C-150000}{150000}=2$.

$\ln\frac{4(C-150000)}{C-600000}=2$.

$\frac{4(C-150000)}{C-600000}=e^2$.

$4C-600000=Ce^2-600000e^2$.

$C=\frac{600000(e^2-1)}{e^2-4}$

$\simeq 1\,130\,000$.

(c) (i) $\underset{\sim}{v}.\underset{\sim}{v}=|\underset{\sim}{v}||\underset{\sim}{v}|\cos 0^\circ=|\underset{\sim}{v}|^2$.

(ii) $\overrightarrow{AC}=\underset{\sim}{a}+\underset{\sim}{b}, \overrightarrow{BD}=\overrightarrow{BA}+\overrightarrow{AD}=k\underset{\sim}{b}-\underset{\sim}{a}$.

$|\overrightarrow{AC}|=|\overrightarrow{BD}|, \therefore |\underset{\sim}{a}+\underset{\sim}{b}|=|k\underset{\sim}{b}-\underset{\sim}{a}|$.

$|\underset{\sim}{a}+\underset{\sim}{b}|^2=|k\underset{\sim}{b}-\underset{\sim}{a}|^2$.

$(\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b})=(k\underset{\sim}{b}-\underset{\sim}{a})(k\underset{\sim}{b}-\underset{\sim}{a})$, using (i).

$\underset{\sim}{a}.\underset{\sim}{a}+2\underset{\sim}{a}.\underset{\sim}{b}+\underset{\sim}{b}.\underset{\sim}{b}=k^2\underset{\sim}{b}.\underset{\sim}{b}-2k\underset{\sim}{a}.\underset{\sim}{b}+\underset{\sim}{a}.\underset{\sim}{a}$.

$2\underset{\sim}{a}.\underset{\sim}{b}+|\underset{\sim}{b}|^2=k^2|\underset{\sim}{b}|^2-2k\underset{\sim}{a}.\underset{\sim}{b}$, using (i).

$2\underset{\sim}{a}.\underset{\sim}{b}(1+k)+(1-k^2)|\underset{\sim}{b}|^2=0$.

$2\underset{\sim}{a}.\underset{\sim}{b}(1+k)+(1-k)(1+k)|\underset{\sim}{b}|^2=0$.

$\therefore 2\underset{\sim}{a}.\underset{\sim}{b}+(1-k)|\underset{\sim}{b}|^2=0$, since $1+k\neq 0$.

(d) $\mu = p = \frac{3}{500} = 0.006, \hat{p} = \frac{4}{500} = 0.008.$

$\sigma^2 = \frac{p(1-p)}{n} = \frac{0.006 \times 0.994}{n} = \frac{0.005964}{n}.$

Converting 0.008 to z score,

$z = \frac{0.008 - 0.006}{\sqrt{\frac{0.005964}{n}}} = 0.0259\sqrt{n}.$

$P(X \geq 2.5\%)$ corresponds to $P(z \geq 2)$,

$0.0259\sqrt{n} \geq 2.$

$n \geq \frac{2^2}{0.0259^2} = 5962.$

$\therefore$ To the nearest thousand, $n = 6000$.

(e) Let $f(x) = xg^{-1}(x)$

$\frac{df(x)}{dx} = g^{-1}(x) + x\frac{d}{dx}g^{-1}(x).$

Since the gradient of the function at (x, y) is the reciprocal of the gradient of its inverse at (y, x), for $g'(x) = 3x^2 + 4$, at $(1,3), g'(1) = 7, \therefore$ the gradient of $g^{-1}(x)$ at $(3,1)$ is $\frac{1}{7}$.

$\therefore$ At $(3,1)$, $\frac{df(x)}{dx} = 1 + 3 \times \frac{1}{7} = \frac{10}{7}.$